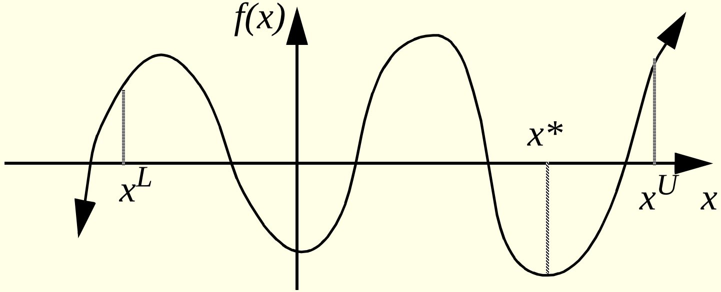
- Stationary Points and Local and Global Minimum
  - Global (Absolute) Minimum



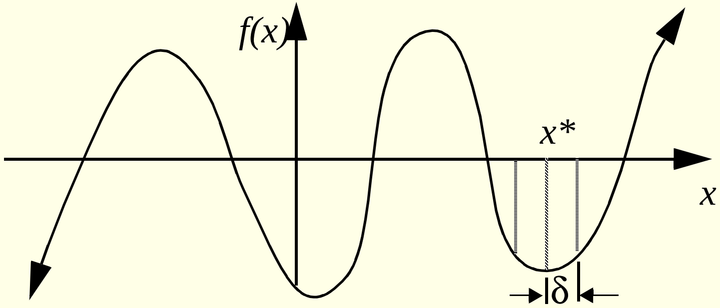
$f^* = f(x^*) \leq f(x)$

for all $x^L \leq x \leq x^U$

strict global minimum

$f^* = f(x^*) < f(x)$

  - Local (Relative) Minimum



$f^* = f(x^*) \leq f(x)$

for $\| x - x^* \| < \delta$

- Necessary and Sufficient Conditions for Optimality:
  - The conditions that ***must*** be satisfied at the optimum point are called necessary

    If any point does not satisfy the necessary condition, it cannot be optimum

    However, not every point that satisfy the necessary conditions are optimal
  - Points satisfying the necessary conditions are called ***candidate optimum points***.
  - If a candidate optimum point satisfies the sufficient condition, then it is indeed optimum.
  - If sufficiency conditions cannot be used or they are not satisfied, we may not be able to draw any conclusions about the optimality of the candidate point.

UNCONSTRAINED PROBLEMS

- Optimality Conditions for a Function of Single Variable:
  - Let x^* be the minimum point, and investigate its neighborhood (i.e., points x at a small distance d from x^*).

$$f(x) - f(x^*) = \Delta f(x) \geq 0$$

  - Based on Taylor series expansion

$$\Delta f(x) = f'(x^*)d + \frac{1}{2}f''(x^*)d^2 + R \geq 0$$

  - For small d, the term $f'(x^*)d$ dominates the series, therefore

$$f'(x^*)d \geq 0$$

  - Since d can arbitrarily take any sign

$$f'(x^*) = 0$$ **NECESSARY CONDITION**

- Sufficiency Conditions for a Function of Single Variables:
  - If the necessary condition is satisfied, then

$$\Delta f(x) = \frac{1}{2} f''(x^*) d^2 + R \geq 0$$

  - Now, the term $f''(x^*)d^2$ dominates the series, therefore

$$f''(x^*) d^2 \geq 0$$

  - Since d^2 is always positive regardless of the sign of d

$$f''(x^*) > 0$$ **SUFFICIENT CONDITION**

IN GENERAL: the lowest nonzero derivative must be even ordered for stationary points (necessary conditions), and it must be positive for local minimum points (sufficiency condition). All odd ordered derivatives lower than the nonzero even ordered derivative must be zero as the necessary condition.

- Optimality Conditions for Functions of Several Variables:
  - The necessary condition

$$\nabla f(\boldsymbol{x}^*) = \mathbf{0} \qquad \frac{\partial f(\boldsymbol{x}^*)}{\partial x_i} = 0 \qquad i = 1, \ldots, n$$

stationary points

  - Now, the second order term dominates the series, therefore

$$\boldsymbol{d}^T \bullet \boldsymbol{H}(\boldsymbol{x}^*) \bullet \boldsymbol{d} > 0$$

i.e., Hessian *H* is positive definite